Finite Element Time Domain Simulation of Electric Discharges

Alexandre A. Costa, Wilson A. Artuzi Jr., *Member, IEEE*, and Marlio J. C. Bonfim

Abstract— A formulation to deal with materials having dispersive and nonlinear characteristics in the finite element time domain method is proposed. The approach is illustrated by an electric discharge simulation in a spark gap switch based on the collision plasma vector differential equation with a polarization current dependent resistivity. The simulated voltage response across the spark gap switch matches the waveform of the measured curve and allows the extraction of the unknown plasma frequency for the electric discharge in air.

Keywords— finite element, nonlinear vector wave equation, electric discharge, plasma

I. INTRODUCTION

Numerical simulation of electric discharges have been successfully implemented by hybrid methods coupling the mass continuity equation with Poisson's equation of electrostatics by means of the flux-corrected transport scheme [1], for instance. The simulation of microwave induced plasma and unwanted radio frequency noise generated by corona effects, however, cannot be treated by Poisson's equation because it is not able to predict electromagnetic wave propagation. In these cases, the Helmholtz wave equation should be used instead. Time domain methods for solving the electric field vector wave equation have been recognized as powerful tools to simulate electrodynamic phenomena [2]. Numerous techniques have been applied to the finite difference time domain method to solve wave propagation problems involving dispersive media [3] and more recently, with media having both dispersive and nonlinear characteristics [4]. In comparison with the finite difference time domain method, the finite element time domain method has two major advantages: it can deal with complex geometries due to the unstructured nature of the space discretization and it can eliminate the maximum time step constraint when using the Newmark Beta integration scheme [5],[6]. Nevertheless, a formulation for this method that includes both dispersive and nonlinear material characteristics is still to be developed. The objective of this paper is to simulate an electric discharge in a spark gap device [7] using a formulation for the finite element time domain method that can accomplish both dispersive and nonlinear macroscopic behavior of plasma. The proposed formulation is based on the motion equation of the unmagnetized collision plasma [3] in connection with the finite element method equations. Measurements have been used to extract the plasma frequency of the electric discharge by comparison with numerical results.

W. A. Artuzi Jr. and M. J. C. Bonfim are with Department of Electrical Engineering, Parana Federal University, Curitiba, PR, Brazil, artuzi@eletrica.ufpr.br and marliob@eletrica.ufpr.br.

II. FORMULATION

A. Field Equations

The inhomogeneous Helmholtz equation for the electric field $\vec{E}$ can be written as

$$\varepsilon_r \varepsilon_o \frac{\partial}{\partial t} \vec{E} + \frac{1}{\mu_o} \int \nabla \times \nabla \times \vec{E} dt = -\vec{J}_e - \vec{J}_p \quad (1)$$

where ε_o and μ_o are the free space electric permittivity and magnetic permeability, respectively, ε_r is the dielectric constant, $\vec{J}_e$ is the excitation current density and $\vec{J}_p$ is the plasma polarization current density. The unmagnetized collision plasma equation

$$\frac{1}{\varepsilon_o \omega_p^2} \frac{\partial}{\partial t} \vec{J}_p + \rho \vec{J}_p = \vec{E} \quad (2)$$

establishes the vector relationship between the polarization current and electric field in terms of the plasma frequency ω_p and the plasma resistivity $\rho = \nu / \varepsilon_o \omega_p^2$ which by the way is a function of the electron collision frequency ν [3].

To simulate the electric discharge disruption, the nonlinear behavior must be incorporated. If the plasma resistivity is very high it can represent a weakly ionized gas as $\vec{J}_p \rightarrow 0$, otherwise, it will represent the gas ionization process as $\vec{J}_p$ increases. The hyperbolic tangent is a suitable function to implement a smooth transition between the two states as

$$\rho = \frac{\rho_{\min} + \rho_{\max}}{2} + \frac{\rho_{\min} - \rho_{\max}}{2} \tanh \left[\frac{\pi}{2} \left(\frac{|\vec{J}_p|}{J_t} - 1 \right) \right] \quad (3)$$

where $\rho_{\max}$ is the high resistivity of the weakly ionized gas, $\rho_{\min}$ is the fully ionized gas resistivity and J_t adjusts the transition point.

B. Decoupling

Decoupling of equations (1) and (2) is desirable because solving them simultaneously becomes a hard computer task due to the nonlinearity imposed to (2). It has been observed experimentally that the addition of the linear term $\int \varepsilon_o \omega_p^2 \vec{E} dt$ to both sides of (1) preserves the numerical stability when the two equations are solved independently within a time step. Thus, the finite element method will be applied to

$$\begin{aligned} \varepsilon_r \varepsilon_o \frac{\partial}{\partial t} \vec{E} + \frac{1}{\mu_o} \int \nabla \times \nabla \times \vec{E} dt + \int \varepsilon_o \omega_p^2 \vec{E} dt = \\ = -\vec{J}_e - -\vec{J}_p + \int \varepsilon_o \omega_p^2 \vec{E} dt. \end{aligned} \quad (4)$$

The reason for the numerical stability preservation may rely upon the fact the eigenspectrum of the finite element form of (4) is expanded to contain those of the finite element forms of (1) and (2) [5].

C. Finite Element Method

The spatial discretization of the vector wave equation follows the conventional expansion of the electric field inside a finite element as

$$\vec{E} = -\sum_j v_j \vec{W}_j \quad (5)$$

where $\vec{W}_j$ are Whitney vector basis functions associated with the element edges and the time dependent scalars v_j are voltages along the corresponding edges [8]. The application of the Galerkin procedure allows (4) to be cast as the electric circuit matrix equation

$$C \frac{d}{dt} v + (K + Q) \int v \, dt = i_e + i_p + Q \int v \, dt \quad (6)$$

where v and i_e are column vectors which contain the unknown voltages and the known excitation currents on the edges of the element, C , K and Q are symmetric square matrices whose entries are given by

$$C_{jk} = \int_V \epsilon_r \epsilon_o \vec{W}_j \cdot \vec{W}_k \, dV \quad (7)$$

$$K_{jk} = \int_V \frac{1}{\mu_o} \nabla \times \vec{W}_j \cdot \nabla \times \vec{W}_k \, dV \quad (8)$$

$$Q_{jk} = \int_V \epsilon_o \omega_p^2 \vec{W}_j \cdot \vec{W}_k \, dV \quad (9)$$

and i_p is a column vector with

$$i_{pk} = \int_V \vec{J}_p \cdot \vec{W}_k \, dV \quad (10)$$

being V the volume of the element. The global wave equation is achieved by performing the superposition of C , K , Q and i_p entries when all the edges are addressed under a global numbering procedure.

D. Linear Time Stepping

The discretization in time is based on the Newmark Beta technique with $\beta = 1/4$. Following the procedure described in [6], equation (6) can be approximated by the set of recursive equations

$$\left[C + \frac{\Delta t^2}{4} (K + Q) \right] u^n = i_e^n + i_p^n - K \, w^n \quad (11)$$

$$v^{n+1/2} = v^{n-1/2} + \Delta t \, u^n \quad (12)$$

$$w^{n+1} = w^n + \Delta t \, v^{n+1/2} \quad (13)$$

where Δt is the time step and the index n represents the time instant $n\Delta t$. The system of linear equations (11) can be efficiently solved by the conjugate gradient method with diagonal preconditioning [6]. The desired voltages on the edges of the elements are updated in (12). The update of i_p , however, depends on the solution of the nonlinear differential equation (2).

E. Nonlinear Time Stepping

The great advantage of the formulation above is the fact the plasma nonlinear vector equation can be solved apart for each element of the spatial discretization where the electric field and polarization current can be assumed to be spatially constant inside each element, but time dependent. This is useful in parallel processing implementation. For simplicity, only two vector components will be considered, but the extension to three components is straightforward. Time discretization of (2) is conducted by the trapezoidal integration scheme which gives

$$\begin{bmatrix} X \\ Y \end{bmatrix}_m = \frac{1}{\epsilon_o \omega_p^2 \Delta t} \left(\begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_m^{n+1} - \begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_m^n \right) + \frac{\rho_m^{n+1}}{2} \begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_m^{n+1} + \frac{\rho_m^n}{2} \begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_m^n - \begin{bmatrix} E_x \\ E_y \end{bmatrix}_m^{n+1/2}. \quad (14)$$

An iterative process represented by the index m must be applied to (14) in order to find the new values of J_{px} and J_{py} that minimize the residuals X and Y . Following [4], the Newton's method given by

$$\begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_{m+1}^{n+1} = \begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_{m+1}^{n+1} - \begin{bmatrix} \frac{\partial X}{\partial J_{px}} & \frac{\partial X}{\partial J_{py}} \\ \frac{\partial Y}{\partial J_{px}} & \frac{\partial Y}{\partial J_{py}} \end{bmatrix}_m^{-1} \begin{bmatrix} X \\ Y \end{bmatrix}_m \quad (15)$$

is employed taking

$$\begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_1^{n+1} = \begin{bmatrix} J_{px} \\ J_{py} \end{bmatrix}_1^n \quad (16)$$

as initial guesses for the first iteration. This iterative process has to be repeated every new time step. The derivatives of the residuals take into account only the new values of J_{px} and J_{py} as

$$\left[\frac{\partial X}{\partial J_{px}} \right]_m = \frac{1}{\Delta t} + \frac{\epsilon_o \omega_p^2}{2} \left[\rho + J_{px} \frac{\partial \rho}{\partial J_{px}} \right]_m^{n+1} \quad (17)$$

$$\left[\frac{\partial X}{\partial J_{py}} \right]_m = \frac{\epsilon_o \omega_p^2}{2} \left[J_{px} \frac{\partial \rho}{\partial J_{py}} \right]_m^{n+1} \quad (18)$$

$$\left[\frac{\partial Y}{\partial J_{px}} \right]_m = \frac{\epsilon_o \omega_p^2}{2} \left[J_{py} \frac{\partial \rho}{\partial J_{px}} \right]_m^{n+1} \quad (19)$$

$$\left[\frac{\partial Y}{\partial J_{py}} \right]_m = \frac{1}{\Delta t} + \frac{\epsilon_o \omega_p^2}{2} \left[\rho + J_{py} \frac{\partial \rho}{\partial J_{py}} \right]_m^{n+1} \quad (20)$$

Equation (5) is used to evaluate $\vec{E}^{n+1/2}$ in the barycenter of each element and the time step calculation is completed using (10) to compute i_p^{n+1} for the next step in (11).

III. EXPERIMENT

A. Device

A spark gap switch [7] has been constructed in order to obtain the plasma parameters of the electric discharge in

Fig. 1. Cross-sectional view of the finite element spark gap switch model.

air. The device consists of four high-voltage 1 nF capacitors sandwiched between two metallic disks and a metallic rod with a rounded end placed in the center of one disk to provide an air filled region of increased electric field intensity. The disks are fed by a continuous electric current source of $50 \mu\text{A}$ and the voltage response between the two plates is monitored by a LeCroy LT584 digital storage oscilloscope through a resistive voltage attenuator.

B. Model

The numerical model takes the advantage of the axial symmetry of the device, thus a 2D simulation can be carried out. The spark gap switch is a body of revolution whose axial and radial axes are identified in the finite element model, shown in Fig. 1, by y and x , respectively. From equation (7) to (10), the volume of the element has to be replaced by $V = 2\pi x_o S$ to take into account the axial symmetry, being x_o the radial distance of the triangle barycenter and S the triangle area. A triangular mesh with 735 elements having an average edge size of 0.23 mm is used for the spatial discretization. The mesh is refined in the region of the rod tip where the polarization current density is supposed to be higher. The metallic surfaces are modelled as perfect electric conductors by imposing $v_j = 0$ on the mesh edges that lay on them. A dielectric constant of $\epsilon_r = 46000$ is used to model the additional 4 nF capacitance placed in the outer region of the device while $\epsilon_r = 1$ is used in the inner region filled with air. In the simulation, the excitation current source and the voltage response are both applied and sampled on the edges of the outer border as depicted in Fig. 1.

C. Plasma Parameters

The plasma parameters $\rho_{\max}$, $\rho_{\min}$, J_t and ω_p need to be carefully estimated to provide a realistic simulation of the electric discharge. The parameter $\rho_{\max}$ is the resistivity of the weakly ionized air which cannot be determined accurately due to the low level of random ionization caused by high frequency electromagnetic waves, as

Fig. 2. Static relationship between the intensities of electric field and polarization current density for $\rho_{\max} = 10^{12} \Omega\text{m}$, $\rho_{\min} = 7.10^{-3} \Omega\text{m}$ and $J_t = 6 \mu\text{A}/\text{m}^2$.

light for example. Therefore, its value has been assumed to be $\rho_{\max} = 10^{12} \Omega\text{m}$ which is in the range of common porous isolating materials. The value $\rho_{\min} = 7.10^{-3} \Omega\text{m}$ has been chosen as the minimum value measured in thermally ionized air according to the experiments performed in [9]. The parameter J_t adjusts the transition between weakly and fully ionized states, but it also defines the dielectric strength $E_{\max} \approx \rho_{\max} J_t / 2$. In Fig. 2, it is shown the static relationship between the intensities of electric field and polarization current density given by (2) with $\partial \vec{J}_p / \partial t = 0$ for $\rho_{\max} = 10^{12} \Omega\text{m}$, $\rho_{\min} = 7.10^{-3} \Omega\text{m}$ and $J_t = 6 \mu\text{A}/\text{m}^2$ where $E_{\max} \approx 3 \text{ MV}/\text{m}$ represents the dielectric strength of air. An estimate for the parameter ω_p could not be found in literature, thus a range of suitable values has been approached by trial and error in comparison with the measured result.

D. Results

Transient voltage responses were simulated with the parameters defined above and with $\omega_p = 30$ and 35 rad/ns . The low intensity continuous current source of $50 \mu\text{A}$ used in the practical experiment produces a small voltage slope of $12 \mu\text{V}/\text{ns}$ before full ionization. Though it is suitable for practical purposes, it is inadequate for simulations due to the extremely long time that would be required until the discharge threshold is reached. This drawback has been overcome in the simulations by adding to the $50 \mu\text{A}$ source a high intensity pulsed current source having a peak of 710 A and a width of 30 ns, as shown in Fig. 3. The peak value is high enough for the discharge disruption to be reached in a shorter time and the width is adjusted to turn off this additional source before the disruption process starts, otherwise, it would distort the desired voltage response. A time step of $\Delta t = 1 \text{ ps}$ is used. This is five times larger than it would be needed in a finite difference time domain simulation with the same spatial grid size. Time stepping

Fig. 3. Waveform of the excitation current used to simulate the electric discharge.

Fig. 4. Measured (---) and simulated (—) voltage transient responses across the spark gap switch terminals for $\omega_p = 30$ rad/ns.

is processed with 10 conjugate gradient iterations and three Newton's iterations per time step. Increasing the number of iterations does not improve the response accuracy significantly. As shown in Figs. 4 and 5, the responses have the shape of a damped cosine function. The measured curve has been time shifted to match the first zero crossing of the simulated curve to facilitate comparison. The difference in the breakdown voltage resultant from simulated and measured curves arises mostly from rod tip geometrical imperfections of the practical device. It is possible to notice that both dumping factor and oscillation frequency of the simulated response increase with ω_p whose optimum value lay between 30 and 35 rad/ns.

Fig. 6 shows three simulated snapshots of the time evolution of the resistivity over the device cross-section for $\omega_p = 35$ rad/ns. The ionization process begins at 40 ns,

Fig. 5. Measured (---) and simulated (—) voltage transient responses across the spark gap switch terminals for $\omega_p = 35$ rad/ns.

Fig. 6. Time evolution of resistivity over the device cross-section at the beginning of the electric discharge disruption.

approximately, and reaches a stationary pattern at 42 ns, according to the time scale of the graph shown in Fig. 5.

IV. CONCLUSIONS

Dispersive and nonlinear material characteristics have been incorporated in the unconditionally stable finite element time domain formulation. The approach is illustrated by an electric discharge simulation in a spark gap switch based on the collision plasma vector differential equation with a polarization current dependent resistivity. The nonlinear problem is solved with the Newton's method as it has been applied to weakly nonlinear optical devices in finite difference time domain simulations. Numerical results have shown the proposed approach is suitable for the simulation of strongly nonlinear phenomena as well. The simulated voltage response across the spark gap switch matches the waveform of the measured curve and allows the extraction of the unknown plasma frequency for the electric discharge

in air. A simple equation has been used to model the resistivity transition between weakly and fully ionized states, however, efforts to develop a more realistic representation of the ionization process must be conducted.

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Alexandre A. Costa received the B.Sc. and M.Sc. degrees in Electrical Engineering from Parana Federal University, Brazil, in 2004 and 2007 respectively. His research interests are concerned with RF noise generation by corona effects on high-voltage power transmission lines.

Wilson A. Artuzi Jr. (S'87-M'93) received the B.Sc. degree in Electrical Engineering from Parana Federal University, Brazil, in 1986 and the M.Sc. and Ph.D. degrees from Tohoku University, Japan, in 1990 and 1993, respectively. Since 1993 he is with the Department of Electrical Engineering of Parana Federal University where he is an associate professor. His research interests involve the application of the FETD method in simulations with materials and discrete devices of complex characteristics for microwave use and electromagnetic interference mitigation. Dr. Artuzi is recipient of the 1994 Japan Microwave Prize.

Marlio J. C. Bonfim received the B.Sc. degree in Electrical Engineering from CEFET-Pr, Brazil, in 1989, the M.Sc. degree from UNICAMP, Brazil, in 1991 and the Ph.D. degree from Université Joseph Fourier, France, in 2001. Since 1992 he is with the Department of Electrical Engineering of Parana Federal University where he is an associate professor. His research interests include microelectronics, electric simulation, electronics instrumentation and magnetic field generation and characterization in the nanosecond scale.